SPECTRAL SHIFT FUNCTION OF HIGHER ORDER

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ABSTRACT. This paper resolves affirmatively Koplienko's conjecture of 1984 on existence of higher order spectral shift measures. Moreover, the paper establishes absolute continuity of these measures and, thus, existence of the higher order spectral shift functions. A spectral shift function of order $n \in \mathbb{N}$ is the function $\eta_n = \eta_{n,H,V}$ such that

$$\operatorname{Tr}\left(f(H+V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} \left[f(H+tV)\right]\Big|_{t=0}\right) = \int_{\mathbb{R}} f^{(n)}(t)\, \eta_n(t)\, dt, \tag{0.1}$$

for every sufficiently smooth function f, where H is a self-adjoint operator defined in a separable Hilbert space $\mathbb{H}$ and V is a self-adjoint operator in the n-th Schatten-von Neumann ideal S^n. We show that $\eta_{n,H,V}$ exists, integrable, and

$$\|\eta_n\|_{L^1(\mathbb{R})} \leq c_n \, \|V\|_{S^n}^n \,,$$

for some constant c_n depending only on $n \in \mathbb{N}$. Existence and summability of η_1 and η_2 were established by Krein in 1953 and Koplienko in 1984, respectively, whereas for $n > 2$ the problem was unresolved. Our method is derived from [22]; it also applies to the general semi-finite von Neumann algebra setting of the perturbation theory.

1. INTRODUCTION

The first order spectral shift function η_1 originated from Lifshits' work on theoretical physics [17]. The mathematical theory of this object was founded by Krein in a series of papers, starting with [15]. The spectral shift function has become a fundamental object in perturbation theory. It can also be recognized as the scattering phase [5] and the spectral flow in a non-commutative geometry setting [3]. The original spectral shift function applies only in the case of trace class perturbations (or trace class differences of the resolvents). The modified second order spectral shift function for Hilbert-Schmidt perturbations was introduced by Koplienko in [16]. In 1984, Koplienko also conjectured existence of the higher order

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spectral shift measures ν_n, $n > 2$, for the perturbation V in S^n such that

$$\operatorname{Tr}\left(f(H+V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} \left[f(H+tV)\right]\Big|_{t=0}\right) = \int_{\mathbb{R}} f^{(n)}(t)\, d\nu_n(t). \tag{1.1}$$

In [16], Koplienko offered a proof which contained an unremovable gap (see [13] for details and [7,14] for additional comments and historical information). In this paper, we prove existence and absolute continuity of ν_n. Note that the density of ν_n is the spectral shift function η_n.

Before stating the main result, we need to fix some notation. Throughout the paper, n denotes a natural number. Let C^n denote the space of all n times continuously differentiable complex-valued functions on $\mathbb{R}$, C_c^n the subclass of C^n of compactly supported functions. We also set $C := C^0$ and $C_c := C_c^0$ and let C_b denote the subclass of continuous bounded functions. Let $W_n \subseteq C^n$ be the class of all functions $f \in C^n$ such that the Fourier transform of $f^{(n)}$ is integrable. For $f \in W_n$, we set

$$\|f\|_{W_n} := \int_{\mathbb{R}} \left|\widehat{f^{(n)}}(s)\right| ds.$$

The class W_n includes the functions f for which $f^{(n)}$ and $f^{(n+1)}$ are in $L^2(\mathbb{R})$ [23, Lemma 7]. In particular, $C_c^{n+1} \subset W_n$.

The first and second order spectral shift functions have been introduced in the von Neumann algebra setting as well. Let $M \subseteq B(\mathbb{H})$ be a semifinite von Neumann algebra and τ a normal faithful semifinite trace on M. Let L^α denote the noncommutative L^α-space with respect to (M, τ) and $\mathcal{L}^\alpha$ the τ-Schatten-von Neumann ideal $L^\alpha \cap M$ (see, e.g., [2,21] and references cited therein for basic definitions and facts). The existence of η_1 and η_2 satisfying (0.1), where the standard trace Tr is replaced with τ and V is taken from $\mathcal{L}^1$ and $\mathcal{L}^2$, respectively, is due to [1,8] and [13].

The main result of the paper is stated below.

Theorem 1.1. *Let $n \in \mathbb{N}$. Let H be a self-adjoint operator affiliated with M and let V be a self-adjoint operator in $\mathcal{L}^n$. Denote $H_t := H + tV$, $0 \le t \le 1$, and let $f \in \cap_{k=0}^n W_k$. Denote*

$$\Delta_{n,f}(H,V) := f(H+V) - \sum_{k=0}^{n-1} \frac{1}{k!} \frac{d^k}{dt^k} \left[f(H_t)\right]\Big|_{t=0}. \tag{1.2}$$

Then $\Delta_{n,f}(H,V) \in \mathcal{L}^1$ and there is a unique function $\eta_n = \eta_{n,H,V} \in L^1(\mathbb{R})$ depending only on n, H, V such that

$$\tau\left(\Delta_{n,f}(H,V)\right) = \int_{\mathbb{R}} f^{(n)}(t)\, \eta_n(t)\, dt \tag{1.3}$$

and

$$\|\eta_n\|_1 \leq c_n \|V\|_n^n .$$

The existence of ν_1 via double operator integration techniques was established in [6]. The absolute continuity of ν_1, or, equivalently, the existence of η_1, was established by Krein years before the development of double operator integration by analytic function and approximation theory techniques. Up to date, there is no double operator integral proof of the absolute continuity of η_1. The proof of the existence of η_2 is due to Koplienko; it is a more delicate application of double operator integration. The existence of η_2 is also established in the present paper.

Our proof of Theorem 1.1 (see Section 2) for $n \geq 2$ is based on a new powerful estimate for multiple operator integrals (see Theorem 2.1 in Section 2 and Theorem 5.3 in Section 5), which extends the advances of [22] for the first order case to the higher order ones. The spectral shift function of order n, with $n > 2$, was constructed explicitly in [13,25,26] under the restrictive assumptions $V \in \mathcal{L}^2$ when $M = B(\mathbb{H})$ and $V \in \mathcal{L}^2$, $n = 3$ when M is a general semi-finite von Neumann algebra. The higher order η_n can be expressed recursively via the lower order η_k, $k \leq n$.

The paper is organized as follows. The proof of Theorem 1.1 is given in Section 2. The technically involved auxiliary results are proved in several steps in subsequent sections. A novel approach to multiple operator integrals suitable to the purposes of this paper is discussed in Section 3.

2. Proof of the main result

Our proof is based on the following powerful estimate for the remainder of the Taylor-type approximation in (1.3).

Theorem 2.1. *Let $n \in \mathbb{N}$. Let H be a self-adjoint operator affiliated with M and let V be a self-adjoint operator in $\mathcal{L}^n$. Denote $H_t := H + tV$, $0 \leq t \leq 1$, and let $f \in \cap_{k=0}^n W_k$. Then $\frac{d^n}{dt^n}[f(H_t)] \in \mathcal{L}^1$ and $\Delta_{n,f}(H,V) \in \mathcal{L}^1$ and there are constants c_n and c_n' (depending only on n) such that the estimates*

$$\left|\tau\left(\frac{d^n}{dt^n}[f(H_t)]\right)\right| \leq c_n \left\|f^{(n)}\right\|_\infty \|V\|_n^n \tag{2.1}$$

and

$$\left|\tau\left(\Delta_{n,f}(H,V)\right)\right| \leq c_n' \left\|f^{(n)}\right\|_\infty \|V\|_n^n \tag{2.2}$$

hold. In addition, the mapping $V \mapsto \tau\left(\Delta_{n,f}(H,V)\right)$ *is continuous on* $\mathcal{L}^n$ *uniformly with respect to* f *in* $\left(\cap_{k=0}^n W_k\right) \cap B$*, where* B *is the unit ball of* C^n *taken with the seminorm* $f \mapsto \|f^{(n)}\|_\infty$.

The proof of Theorem 2.1 is given in the following sections. Assuming that Theorem 2.1 holds, we prove Theorem 1.1.

Proof of Theorem 1.1. By the inequality (2.2) applied to all $f \in C_c^{n+1}$, the Riesz representation theorem for a bounded linear functional on the space of continuous functions on a compact set ensures existence of a unique finite real-valued measure ν_n on $\mathbb{R}$, with

$$\|\nu_n\| \le c_n \, \|V\|_n^n, \tag{2.3}$$

and such that

$$\tau\left(\Delta_{n,f}(H,V)\right) = \int_{\mathbb{R}} f^{(n)}(t)\, d\nu_n(t). \tag{2.4}$$

To finish the proof, we need to demonstrate the absolute continuity of ν_n for $n \ge 2$ (absolute continuity of ν_1 was established in [1,8,15]).

Firstly, we assume that $V \in \mathcal{L}^1 \subseteq \mathcal{L}^n$ and, therefore, Theorem 2.1 guarantees that $\Delta_{n,f}(H,V) \in \mathcal{L}^1$. Then, for every $f \in C_c^n$, integration by parts gives

$$\tau\left(\Delta_{n-1,f}(H,V)\right) = \int_{\mathbb{R}} f^{(n-1)}(t)\, d\nu_{n-1}(t) = -\int_{\mathbb{R}} f^{(n)}(t)\nu_{n-1}((-\infty,t))\, dt. \tag{2.5}$$

By (2.1), we have

$$\left|\tau\left(\frac{d^{n-1}}{dt^{n-1}}\left[f(H_t)\right]\right)\right| \le c_{n-1}\left\|f^{(n-1)}\right\|_\infty \|V\|_{n-1}^{n-1}, \ \ f \in C_c^n,$$

and thus the Riesz representation theorem implies the existence of a unique finite measure μ_{n-1} on $\mathbb{R}$ such that

$$\begin{aligned}\frac{1}{(n-1)!}\tau\left(\frac{d^{n-1}}{dt^{n-1}}\left[f(H_t)\right]\right) &= \int_{\mathbb{R}} f^{(n-1)}(t)\, d\mu_{n-1}(t)\\ &= -\int_{\mathbb{R}} f^{(n)}(t)\mu_{n-1}((-\infty,t))\, dt.\end{aligned} \tag{2.6}$$

By combining (2.5) and (2.6) in (1.2), we obtain

$$\tau\left(\Delta_{n,f}(H,V)\right) = \int_{\mathbb{R}} f^{(n)}(t)\left[\mu_{n-1}((-\infty,t)) - \nu_{n-1}((-\infty,t))\right] dt,$$

which along with (2.4) implies that ν_n is absolutely continuous and its density equals

$$\eta_n(t) = \mu_{n-1}((-\infty,t)) - \nu_{n-1}((-\infty,t)).$$

Due to (2.3), we have that $\|\eta_n\|_1 \le c_n \|V\|_n^n$. Thus, the existence of the spectral shift function of order n is proved for $V \in \mathcal{L}^1$.

To prove the existence of η_n in the case of a general $V \in \mathcal{L}^n$, we choose a sequence of operators $\{V_k\}_k \subseteq \mathcal{L}^1$ such that $\lim_{k\to\infty} \|V - V_k\|_n = 0$ (see, e.g., [9]). We now show that the sequence of the (integrable) spectral shift functions $\{\eta_{n,H,V_k}\}_k$ is Cauchy in $L^1(\mathbb{R})$. First, by duality, we obtain

$$\begin{aligned}&\int_{\mathbb{R}} \left|\eta_{n,H,V_j}(t) - \eta_{n,H,V_k}(t)\right|\, dt \\ &= \sup_{f\in C_c^n,\ \|f^{(n)}\|_\infty\le 1} \left|\int_{\mathbb{R}} \left(\eta_{n,H,V_j}(t) - \eta_{n,H,V_k}(t)\right) f^{(n)}(t)\, dt\right|.\end{aligned}$$

By (2.4),

$$\left|\int_{\mathbb{R}} \left(\eta_{n,H,V_j}(t) - \eta_{n,H,V_k}(t)\right) f^{(n)}(t)\, dt\right| = \left|\tau\left(\Delta_{n,f}(H,V_j) - \Delta_{n,f}(H,V_k)\right)\right|.$$

The uniform continuity of $V \mapsto \tau\left(\Delta_{n,f}(H,V)\right)$ (see Theorem 2.1) implies

$$\lim_{j,k\to\infty} \int_{\mathbb{R}} \left|\eta_{n,H,V_j}(t) - \eta_{n,H,V_k}(t)\right|\, dt = 0.$$

Thus, the sequence $\{\eta_{n,H,V_k}\}_k$ converges to an integrable function, which we denote by $\eta_{n,H,V}$. Since L^1-norms of the functions η_{n,H,V_k}, $k \in \mathbb{N}$, are uniformly bounded by $c_n\|V\|_n^n$, we obtain

$$\|\eta_{n,H,V}\|_1 \le c_n\|V\|_n^n.$$

By passing to the limit in the representation

$$\tau\left(\Delta_{n,f}(H,V_k)\right) = \int_{\mathbb{R}} f^{(n)}(t)\, \eta_{n,H,V_k}(t)\, dt$$

as $V_k \to V$, we obtain that $\eta_{n,H,V}$ satisfies the trace formula (0.1) and, thus, it is the spectral shift function of order n corresponding to the general perturbation $V \in \mathcal{L}^n$. □

Remark 2.2. The earlier mentioned explicit representation of [13] for η_n along with the summability of η_n implies

$$\int_{\mathbb{R}} \eta_n(t)\, dt = \tau(V^n)/n! \tag{2.7}$$

for V in the Hilbert-Schmidt class [25] and thus, by approximations, for V in the n-th Schatten von Neumann ideal. We remark that the equality (2.7) is a routine calculation in the case of a bounded operator H (for more details see, e.g., [13, Lemma 3.5]). It follows from the trace formula (1.3) applied to a function $f \in W_n$, which coincides with the polynomial t^n on the spectra of all operators involved in (1.3). The property (2.7) in the case of an unbounded operator H can also be

obtained by approximations from the bounded case. The equality (2.7) for $n = 1$ was obtained in [1,15] and for $n = 2$ in [19].

3. Multiple Operator Integrals

Let H be a self-adjoint linear operator affiliated with M and let dE_λ, $\lambda \in \mathbb{R}$ be the corresponding spectral measure. We set $E_{l,m} = E\left[\frac{l}{m}, \frac{l+1}{m}\right)$, for every $m \in \mathbb{N}$ and $l \in \mathbb{Z}$. The symbols H, dE_λ and $E_{l,m}$ will keep their meaning for the rest of the manuscript.

Definition 3.1. Let $n \in \mathbb{N}$ and let $1 \leq \alpha_j \leq \infty$, with $1 \leq j \leq n$, be such that $0 \leq \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} \leq 1$. Let $x_j \in L^{\alpha_j}$ and denote $\tilde{x} := (x_1, \ldots, x_n)$. Fix a bounded Borel function $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$. Suppose that for every $m \in \mathbb{N}$, the series[1]

$$S_{\phi,m}(\tilde{x}) := \sum_{l_0,\ldots,l_n \in \mathbb{Z}} \phi\left(\frac{l_0}{m}, \ldots, \frac{l_n}{m}\right) E_{l_0,m} x_1 E_{l_1,m} x_2 \cdot \ldots \cdot x_n E_{l_n,m}$$

converges in the norm of L^α, where $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n}$ and

$$\tilde{x} \mapsto S_{\phi,m}(\tilde{x}), \ \ m \in \mathbb{N},$$

is a sequence of bounded polylinear operators $L^{\alpha_1} \times \ldots \times L^{\alpha_n} \mapsto L^\alpha$. If the sequence of operators $\left\{S_{\phi,m}\right\}_{m\geq 1}$ converges strongly to some polylinear operator T_ϕ, then, according to the Banach-Steinhaus theorem, $\left\{S_{\phi,m}\right\}_{m\geq 1}$ is uniformly bounded and the operator T_ϕ is also bounded. The operator T_ϕ is called the *multiple operator integral* associated with ϕ and the operator H (or the spectral measure dE_λ). If T_ϕ exists as above and bounded, we shall be simply saying that T_ϕ is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ for brevity.

Throughout the paper, we shall frequently use the following algebraic properties of the mapping $\phi \mapsto T_\phi$. The proof is immediate from the definition above.

Lemma 3.2. *Let $1 \leq \alpha_j \leq \infty$, for $1 \leq j \leq n$, be such that $0 \leq \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} \leq 1$. Let $x_j \in L^{\alpha_j}$, $1 \leq j \leq n$.*

(i) *Let $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$ be bounded Borel and let T_ϕ be bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$. If*

$$\bar{\phi}(\lambda_0, \lambda_1, \ldots, \lambda_n) := \overline{\phi(\lambda_n, \lambda_{n-1}, \ldots, \lambda_0)},$$

then $T_{\bar{\phi}}$ is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ and

$$\left\|T_\phi\right\| = \left\|T_{\bar{\phi}}\right\|.$$

[1] understood as $\lim_{N\to\infty} \sum_{|l_j|\leq N,\, j\leq 0\leq n} \phi\left(\frac{l_0}{m}, \ldots, \frac{l_n}{m}\right) E_{l_0,m} x_1 E_{l_1,m} x_2 \cdot \ldots \cdot x_n E_{l_n,m}$

(ii) *Assume, in addition, that* $1 \leq \alpha_0 \leq \infty$ *and* $\frac{1}{\alpha_0} + \ldots + \frac{1}{\alpha_n} = 1$. *Let* $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$ *be a bounded Borel function. Assume that* T_ϕ *exists and is bounded on* $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$. *Define*

$$\phi^*(\lambda_n, \lambda_0, \ldots, \lambda_{n-1}) := \phi(\lambda_0, \ldots, \lambda_{n-1}, \lambda_n).$$

If T_{ϕ^*} *exists and is bounded on* $L^{\alpha_0} \times \ldots \times L^{\alpha_{n-1}}$, *then* [2]

$$\tau\left(x_0 T_\phi(x_1, \ldots, x_n)\right) = \tau\left(T_{\phi^*}(x_0, \ldots, x_{n-1})\, x_n\right).$$

(iii) *Let* $\phi_1 : \mathbb{R}^{k+1} \mapsto \mathbb{C}$ *and* $\phi_2 : \mathbb{R}^{n-k+1} \mapsto \mathbb{C}$ *be bounded Borel functions such that the operators* T_{ϕ_1} *and* T_{ϕ_2} *exist and are bounded on* $L^{\alpha_1} \times \ldots L^{\alpha_k}$ *and* $L^{\alpha_{k+1}} \times \ldots \times L^{\alpha_n}$, *respectively. If*

$$\psi(\lambda_0, \ldots, \lambda_n) := \phi_1(\lambda_0, \ldots, \lambda_k) \cdot \phi_2(\lambda_k, \ldots, \lambda_n),$$

then the operator T_ψ *exists and is bounded on* $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ *and*

$$T_\psi(x_1, \ldots, x_n) = T_{\phi_1}(x_1, \ldots, x_k) \cdot T_{\phi_2}(x_{k+1}, \ldots, x_n).$$

(iv) *Let* $\phi_1 : \mathbb{R}^{k+1} \mapsto \mathbb{C}$ *and* $\phi_2 : \mathbb{R}^{n-k+2} \mapsto \mathbb{C}$ *be bounded Borel functions such that* T_{ϕ_1} *and* T_{ϕ_2} *exist and are bounded on* $L^{\alpha_1} \times \ldots \times L^{\alpha_k}$ *and* $L^{\alpha} \times L^{\alpha_{k+1}} \times \ldots \times L^{\alpha_n}$, *respectively, where* $\frac{1}{\alpha} = \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_k}$. *If*

$$\psi(\lambda_0, \ldots, \lambda_n) := \phi_1(\lambda_0, \ldots, \lambda_k) \cdot \phi_2(\lambda_0, \lambda_k, \ldots, \lambda_n),$$

then the operator T_ψ *exists and is bounded on* $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ *and*

$$T_\psi(x_1, \ldots, x_n) = T_{\phi_2}\left(T_{\phi_1}(x_1, \ldots, x_k), x_{k+1}, \ldots, x_n\right).$$

The next lemma shows that in dealing with the operators T_ϕ it is always sufficient to consider compactly supported functions ϕ.

[2] Note that we do not imply boundedness of T_{ϕ^*} from that of T_ϕ. This is due to the fact that strong operator topology is not well compatible with duality, i.e., there is an example of a sequence of operators converging strongly such that the sequence of dual operators does not converge.

Lemma 3.3. *Let $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$ be a bounded Borel function and let $1 \leq \alpha_j < \infty$, with $1 \leq j \leq n$, be such that $0 \leq \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} \leq 1$. If*

$$\phi_k\left(\tilde{\lambda}\right) := \begin{cases} \phi\left(\tilde{\lambda}\right), & \text{if } \left|\lambda_j\right| \leq k, \text{for every } 0 \leq j \leq n; \\ 0, & \text{otherwise,} \end{cases}$$

where $\tilde{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$, and if the sequence of the operators $\{T_{\phi_k}\}_{k=1}^{\infty}$ exists and is uniformly bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, then the operator T_ϕ exists and is bounded (with the same norm estimate) on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$.

Proof of Lemma 3.3. Let $F_k = E([-k,k])$ and let $L^\alpha_k = F_k L^\alpha F_k \subseteq L^\alpha$. Observe that if the collection $\tilde{x} = (x_1, \ldots, x_n)$ is such that $x_j \in L^{\alpha_j}_k$, then

$$y_m := S_{\phi,m}(\tilde{x}) = S_{\phi_k,m}(\tilde{x}).$$

Since the operator T_{ϕ_k} exists and is bounded, the sequence $\{y_m\}_{m\geq 1}$ converges in L^α for every $k \geq 1$. Hence, the operator T_ϕ is well defined and bounded on the subspace $\bigcup_{k\geq 1} L^{\alpha_1}_k \times \ldots \times L^{\alpha_n}_k$. Since the set $\bigcup_{k\geq 1} L^{\alpha_1}_k \times \ldots \times L^{\alpha_n}_k$ is norm dense in $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, the operator T_ϕ is well defined on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$. $\square$

Remark 3.4. In the special case $n = 1$, the polylinear operator T_ϕ becomes a linear operator on L^α. In this case, the operator T_ϕ is called the double operator integral [18] (see also references cited in [6,18]). In particular, it is known that on the Hilbert space L^2, the norm of the operator T_ϕ is bounded by $\|\phi\|_\infty$, for every bounded Borel ϕ. Note that this nice Hilbert space behaviour is exclusive to the case $n = 1$. For $n \geq 2$ it seems there is no combination of exponents $\alpha_1, \ldots, \alpha_n$ such that T_ϕ is bounded for every bounded Borel ϕ.

We next introduce the subclass of functions ϕ for which it is relatively simple to show that the operator T_ϕ is bounded, though the functions themselves have rather complex structure.

Let $\mathfrak{C}_n$ be the class of functions $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$ admitting the representation

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega \prod_{j=0}^{n} a_j(\lambda_j, s)\, d\mu(s), \tag{3.1}$$

for some finite measure space (Ω, μ) and bounded continuous functions

$$a_j\left(\cdot, s\right) : \mathbb{R} \mapsto \mathbb{C}$$

for which there is a growing sequence of measurable subsets $\{\Omega_k\}_{k\geq 1}$, with $\Omega_k \subseteq \Omega$ and $\cup_{k\geq 1}\Omega_k = \Omega$ such that the families

$$\left\{a_j(\cdot,s)\right\}_{s\in\Omega_k},\ \ 0\leq j\leq n,\ k\geq 1,$$

are uniformly bounded and uniformly continuous. The class $\mathfrak{C}_n$ has the norm

$$\|\phi\|_{\mathfrak{C}_n} = \inf \int_\Omega \prod_{j=0}^n \left\|a_j(\cdot,s)\right\|_\infty \, d\,|\mu|\,(s),$$

where the infimum is taken over all possible representations (3.1).

The following lemma demonstrates that for $f \in \mathfrak{C}_n$, Definition 3.1 coincides with the one in [2, 20].

Lemma 3.5. *Let $1 \leq \alpha_j \leq \infty$, with $1 \leq j \leq n$, be such that $0 \leq \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} \leq 1$. For every $\phi \in \mathfrak{C}_n$, the operator T_ϕ exists and is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, with*

$$\left\|T_\phi\right\| \leq \|\phi\|_{\mathfrak{C}_n}.$$

Moreover, given the decomposition (3.1) of the function ϕ, the operator T_ϕ can be represented as the Bochner integral

$$T_\phi(x_1, x_2, \ldots, x_n) = \int_\Omega a_0(H,s)\, x_1\, a_1(H,s)\, x_2 \cdot \ldots \cdot x_n a_n(H,s)\, d\mu(s). \tag{3.2}$$

Proof of Lemma 3.5. Due to the choice of functions a_j, the operator

$$\hat{T}_\phi(x_1, x_2, \ldots, x_n) := \int_\Omega a_0(H,s)\, x_1\, a_1(H,s)\, x_2 \cdot \ldots \cdot x_n a_n(H,s)\, d\mu(s) \tag{3.3}$$

is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, with

$$\left\|\hat{T}_\phi\right\| \leq \|\phi\|_{\mathfrak{C}_n}.$$

Let us show that $\hat{T}_\phi$ coincides with the multiple operator integral given by Definition 3.1. Let $H_m := \sum_{l\in\mathbb{Z}} \frac{l}{m} E_{l,m}$. Clearly,

$$\|H - H_m\| \leq \frac{1}{m}. \tag{3.4}$$

By analogy with (3.3), we set

$$\hat{T}_{\phi,m}(x_1, x_2, \ldots, x_n) := \int_\Omega a_0(H_m,s)\, x_1\, a_1(H_m,s)\, x_2 \cdot \ldots \cdot x_n a_n(H_m,s)\, d\mu(s).$$

It can be seen that the operator $\hat{T}_{\phi,m}$ coincides with $S_{m,\phi}$ in Definition 3.1, i.e.,

$$S_{\phi,m}\left(\tilde{x}\right) = \hat{T}_{\phi,m}\left(\tilde{x}\right), \quad \tilde{x} = (x_1, \ldots, x_n). \tag{3.5}$$

Thus, to finish the proof, it suffices to show that

$$\lim_{m\to\infty} \left\|\hat{T}_{\phi,m}(\tilde{x}) - \hat{T}_\phi\left(\tilde{x}\right)\right\|_\alpha = 0, \quad \tilde{x} = \left(x_1, \ldots, x_n\right).$$

To this end, fix $\tilde{x} = (x_1, \ldots, x_n)$, fix $\epsilon > 0$, and fix a number $k_\epsilon \in \mathbb{N}$ such that

$$\int_{\Omega \setminus \Omega_{k_\epsilon}} \prod_{j=0}^{n} \left\| a_j(\cdot, s) \right\|_\infty d\,|\mu|\,(s) < \epsilon. \tag{3.6}$$

We next set

$$\hat{y}_\epsilon := \int_{\Omega_{k_\epsilon}} a_0(H,s)x_1 \cdot \ldots \cdot x_n a_n(H,s)\, d\mu(s) \quad \text{and}$$

$$\hat{y}_{\epsilon,m} := \int_{\Omega_{k_\epsilon}} a_0(H_m,s)x_1 \cdot \ldots \cdot x_n a_n(H_m,s)\, d\mu(s).$$

The estimate (3.6) implies

$$\left\| \hat{T}_\phi(\tilde{x}) - \hat{y}_\epsilon \right\|_\alpha < \epsilon \quad \text{and} \quad \left\| \hat{T}_{\phi,m}(\tilde{x}) - \hat{y}_{\epsilon,m} \right\|_\alpha < \epsilon. \tag{3.7}$$

Since the family of functions $\{a_j(\cdot, s)\}_{s \in \Omega_{k_\epsilon}}$, $0 \le j \le n$, is uniformly continuous, we derive from (3.4) existence of $m_\epsilon \in \mathbb{N}$ such that

$$\|\hat{y}_\epsilon - \hat{y}_{\epsilon,m}\|_\alpha < \epsilon, \quad \text{for any } m > m_\epsilon. \tag{3.8}$$

Combining (3.5), (3.7), and (3.8) implies that for every $\epsilon > 0$, there exists $m_\epsilon \in \mathbb{N}$ such that for every $m > m_\epsilon$,

$$\left\| \hat{T}_\phi(\tilde{x}) - \hat{T}_{\phi,m}(\tilde{x}) \right\|_\alpha \le \left\| \hat{T}_\phi(\tilde{x}) - \hat{y}_\epsilon \right\|_\alpha + \|\hat{y}_\epsilon - \hat{y}_{\epsilon,m}\|_\alpha + \left\| \hat{y}_{\epsilon,m} - \hat{T}_{\phi,m}(\tilde{x}) \right\|_\alpha < 3\epsilon,$$

completing the proof. $\square$

Recall that for $f \in W_n$, the operator derivative $\frac{d^n}{dt^n}[f(H+tV)]$ is given by the integral $T_{f^{[n]}}$ [2, 20] where $f^{[n]}$ is the divided difference. The divided difference of the zeroth order $f^{[0]}$ is the function f itself. Let $\lambda_0, \lambda_1, \ldots \in \mathbb{R}$ and let $f \in C^n$. The divided difference $f^{[n]}$ of order n is defined recursively by

$$f^{[n]}\left(\lambda_0, \lambda_1, \tilde{\lambda}\right) = \begin{cases} \frac{f^{[n-1]}(\lambda_0, \tilde{\lambda}) - f^{[n-1]}(\lambda_1, \tilde{\lambda})}{\lambda_0 - \lambda_1}, & \text{if } \lambda_0 \ne \lambda_1, \\ \frac{d}{d\lambda_1} f^{[n-1]}(\lambda_1, \tilde{\lambda}), & \text{if } \lambda_0 = \lambda_1, \end{cases}$$

where $\tilde{\lambda} = (\lambda_2, \ldots, \lambda_n) \in \mathbb{R}^{n-1}$. It is clear that if $f \in C^n$, then the divided difference $f^{[n]}$ is a continuous function on $\mathbb{R}^{n+1}$. If $f \in W_n$, then $f^{[n]} \in \mathfrak{C}_n$ [2, Lemma 2.3].

4. Demonstration of the method.

The proof of Theorem 2.1 is rather technical, so we choose the following approach. In the present section, we firstly demonstrate the principal techniques of the proof in the simplest case $n = 1$. It was already established in [22], but the method of [22] does not allow extension to the higher dimensions. Here we give

an alternative proof suitable for our present purpose and outline the scheme of the proof for $n \geq 2$. The detailed proof of Theorem 2.1 will be given in the next section.

The case $n = 1$. To make the demonstration simpler, we assume that the spectrum of the operator H is concentrated in the integral points, i.e., if dE_λ, $\lambda \in \mathbb{R}$ is the spectral measure of H, then

$$E(B) = \sum_{l \in B \cap \mathbb{Z}} E_l, \ \ B \subseteq \mathbb{R} \text{ is Borel,}$$

where $E = \{E_l\}_{l \in \mathbb{Z}}$ is a family of pairwise orthogonal projections and $\sum_{l \in \mathbb{Z}} E_l = 1$. The operator $T_{f^{[1]}}$ associated with H is given by the multiple sum

$$T_{f^{[1]}}(x) = \sum_{l,m \in \mathbb{Z}} f^{[1]}(l,m)\, E_l x E_m.$$

We shall also consider $f^{[1]}$ only on compact subsets (as in Lemma 3.3); however, we shall not reflect this in our notations. The latter enables to replace double operator integrals with finite sums. Note also that the estimates we present below do not depend on the support of $f^{[1]}$.

Theorem 4.1 ([22, Theorem 2]). *If $f \in C^1$ and $\|f'\|_\infty < \infty$, then the operator $T_{f^{[1]}}$ is bounded on every L^α, with $1 < \alpha < \infty$, and*

$$\left\| T_{f^{[1]}} \right\| \leq c_\alpha \left\| f' \right\|_\infty,$$

where the constant c_α depends only on α.

The proof of Theorem 4.1 is based on the following lemmas.

Lemma 4.2 ([22, Lemma 5]). *There is a function $g : \mathbb{R} \mapsto \mathbb{C}$ such that*

$$\int_{\mathbb{R}} |s|^n \, |g(s)| \, ds < +\infty, \ \ n \geq 0,$$

and such that, for every $\mu \geq \lambda > 0$,

$$\frac{\lambda}{\mu} = \int_{\mathbb{R}} g(s)\, \lambda^{is} \mu^{-is}\, ds.$$

Lemma 4.3 ([22, Lemma 4]). *Let $x \in L^\alpha$, with $1 < \alpha < \infty$. Then*

$$x_s = \sum_{l<m} (m-l)^{is}\, E_l x E_m, \quad s \in \mathbb{R},$$

is well defined and there is a constant $c_\alpha > 0$ such that

$$\|x_s\|_\alpha \leq c_\alpha \ (1 + |s|) \ \|x\|_\alpha \, .$$

The principal step toward Theorem 4.1 is the following lemma.

Lemma 4.4. *Let $2 < \alpha, \beta < \infty$ be such that $2^{-1} = \alpha^{-1} + \beta^{-1}$. Then there is a constant $c_\alpha > 0$ such that, for every $f \in C^1$ with $\|f'\|_\infty \leq 1$,*

$$\left\| T_{f^{[1]}} \right\|_\alpha \leq c_\alpha \left(1 + \left\| T_{f^{[1]}} \right\|_\beta \right),$$

where $\left\| T_{f^{[1]}} \right\|_\alpha$ is the norm of the operator $T_{f^{[1]}} : L^\alpha \mapsto L^\alpha$.

Throughout the text, we agree that the constant symbols c_α are allowed to vary from line to line, or even within a line.

Proof of Lemma 4.4. Let $x \in L^\alpha$ and let $y \in L^{\alpha'}$ where α' is the conjugate exponent, i.e., $\alpha^{-1} + \alpha'^{-1} = 1$. We shall prove the estimate

$$\left| \tau \left(y T_{f^{[1]}}(x) \right) \right| \leq c_\alpha \left(1 + \left\| T_{f^{[1]}} \right\|_\beta \right) \|x\|_\alpha \|y\|_{\alpha'},$$

for some constant $c_\alpha > 0$, which immediately implies the claim of the lemma.

Let us fix $x \in L^\alpha$ and $y \in L^{\alpha'}$. Without loss of generality, we may assume that $\|x\|_\alpha = \|y\|_{\alpha'} = 1$. The triangular truncation is a bounded linear operator on L^α, $1 < \alpha < \infty$ (see, e.g., [12]). Thus, we may further assume that the operator x is upper-triangular and y is lower-triangular with respect to the family $\{E_l\}_{l \in \mathbb{Z}}$.[3] We also may assume that x is off-diagonal as $T_{f^{[1]}}$ is trivially bounded on the diagonal subspace of L^α.

We can assume that y has τ-finite left and right supports because the class of lower-triangular operators with τ-finite supports is norm dense in the lower-triangular part of $L^{\alpha'}$. Let us fix $\epsilon > 0$. Since $\frac{1}{\alpha'} = \frac{1}{2} + \frac{1}{\beta}$, there is a factorization $y = ab$, where $a \in L^2$ and $b \in L^\beta$ are lower-triangular and

$$1 \leq \|a\|_2 \|b\|_\beta \leq 1 + \epsilon. \tag{4.1}$$

Such factorization always exists due to [21, Theorem 8.3].

For every element $z \in M$, we set $z_{lm} := E_l z E_m$ for brevity. Since x is upper triangular and y is lower triangular,

$$\tau \left(y T_{f^{[1]}}(x) \right) = \tau \left(ab T_{f^{[1]}}(x) \right) = \sum_{\substack{l \leq k \leq m \\ l \neq m}} f^{[1]}(l,m) \tau \left(a_{mk} b_{kl} x_{lm} \right).$$

Observing the straightforward decomposition

$$f^{[1]}(l,m) = \frac{l-k}{l-m} f^{[1]}(l,k) + \frac{k-m}{l-m} f^{[1]}(k,m), \quad l \leq k \leq m,\ l \neq m, \tag{4.2}$$

[3] An element $x \in M$ is called *upper-triangular* with respect to a family of pairwise orthogonal projections $\{E_l\}_{l \in \mathbb{Z}}$ if and only if $E_l x E_m = 0$ for every $l > m$; it is called *lower-triangular* if and only if x^* is upper-triangular.

we obtain

$$\tau\left(yT_{f^{[1]}}(x)\right) = \sum_{l<k\leq m} \frac{l-k}{l-m}\,\tau\left(a_{mk}\left(T_{f^{[1]}}(b)\right)_{kl} x_{lm}\right) + \sum_{l\leq k<m} \frac{k-m}{l-m}\,\tau\left(\left(T_{f^{[1]}}(a)\right)_{mk} b_{kl}x_{lm}\right) =: S_1 + S_2.$$

We shall estimate the term S_1. The estimate for the term S_2 can be obtained similarly. Observe that Lemma 4.2 yields the representation

$$\frac{l-k}{l-m} = \int_{\mathbb{R}} g(s)\ (k-l)^{is}\,(m-l)^{-is}\,ds,\ \ l\leq k<m,$$

where $g:\mathbb{R}\mapsto\mathbb{C}$ is such that

$$\int_{\mathbb{R}} |s|^n\ |g(s)|\ ds < +\infty,\ \ n\geq 0. \tag{4.3}$$

Thus, if we set (as in Lemma 4.3)

$$x_s := \sum_{l<m} (m-l)^{is}\ x_{lm}\ \text{ and }\ \left(T_{f^{[1]}}(b)\right)_s := \sum_{l<k} (k-l)^{is}\ \left(T_{f^{[1]}}(b)\right)_{kl},\ \ s\in\mathbb{R},$$

then, by Lemma 3.2 (iii),

$$S_1 = \int_{\mathbb{R}} g(s)\,\tau\left(a\left(T_{f^{[1]}}(b)\right)_s x_{-s}\right)\,ds.$$

Subsequent application of the noncommutative Hölder inequality, Lemma 4.3 to both $\left(T_{f^{[1]}}(b)\right)_s$ and x_s, and the estimate (4.1) implies

$$\begin{aligned}
\left|\tau\left(a\left(T_{f^{[1]}}(b)\right)_s x_{-s}\right)\right| &\leq \|a\|_2\ \left\|\left(T_{f^{[1]}}(b)\right)_s\right\|_\beta\ \|x_{-s}\|_\alpha \\
&\leq\ c_\alpha\ (1+|s|)^2\ \|a\|_2\ \left\|T_{f^{[1]}}(b)\right\|_\beta \\
&\leq c_\alpha\ (1+|s|)^2\ \left\|T_{f^{[1]}}\right\|_\beta\ \|a\|_2\ \|b\|_\beta \\
&\leq c_\alpha\ (1+|s|)^2\ \left\|T_{f^{[1]}}\right\|_\beta\ (1+\epsilon).
\end{aligned}$$

From (4.3), we derive

$$|S_1|\leq\ c_\alpha\,(1+\epsilon)\ \left\|T_{f^{[1]}}\right\|_\beta \int_{\mathbb{R}} (1+|s|)^2\ |g(s)|\ ds\leq\ c_\alpha\ (1+\epsilon)\ \left\|T_{f^{[1]}}\right\|_\beta.$$

Observing that $\epsilon>0$ was arbitrary, we finally arrive at

$$|S_1|\leq\ c_\alpha\ \left\|T_{f^{[1]}}\right\|_\beta.$$

The estimate of the term S_2 is even simpler due to the fact that the element a belongs to the Hilbert space L^2 and, therefore, the factor $\left\|T_{f^{[1]}}\right\|_2$ does not exceed $\|f'\|_\infty\leq 1$ by Remark 3.4. In other words, we have

$$|S_2|\leq\ c_\alpha.$$

Combining the estimates for S_1 and S_2, we finally obtain

$$\left|\tau\left(yT_{f^{[1]}}(x)\right)\right| \leq c_\alpha \left(1 + \left\|T_{f^{[1]}}\right\|_\beta\right).$$

The lemma is completely proved. □

Proof of Theorem 4.1. Without loss of generality we assume that $\|f'\|_\infty \leq 1$. Due to complex interpolation and duality, it is sufficient to give the proof when α is "close" to ∞.

Fix $2 < \alpha < \infty$ sufficiently large and fix $2 < \beta < \infty$ such that $\frac{1}{2} = \frac{1}{\alpha} + \frac{1}{\beta}$. The exponent β approaches 2 as $\alpha \to \infty$. Now, on one hand, Lemma 4.4 gives

$$\left\|T_{f^{[1]}}\right\|_\alpha \leq c_\alpha \left(1 + \left\|T_{f^{[1]}}\right\|_\beta\right)$$

and, on the other hand, the complex interpolation method [4,11] provides [4]

$$\left\|T_{f^{[1]}}\right\|_\beta \leq \left\|T_{f^{[1]}}\right\|_\alpha^\theta, \text{ where } \theta = \frac{2^{-1} - \beta^{-1}}{2^{-1} - \alpha^{-1}}.$$

The latter two inequalities, combined together, imply

$$\left\|T_{f^{[1]}}\right\|_\alpha \leq c_\alpha,$$

completing the proof. □

The case $n \geq 2$. This is reduced to the case $n = 1$. The reduction is based on the following representation for $f^{[2]}$:

$$f^{[2]}(l,k,m) = \frac{l-k}{l-m}\,\phi_2(l,k) + \frac{k-m}{l-m}\,\phi_2(m,k),$$
$$\text{where} \quad \phi_2(l,m) = \frac{f^{[1]}(l,m) - f'(m)}{l-m}. \quad (4.4)$$

For simplicity assume that $l \leq k \leq m$ and $l \neq m$. To see the case $n = 2$, we have to prove that the multiple operator integrals for both functions

$$\frac{l-k}{l-m}\,\phi_2(l,k) \text{ and } \frac{k-m}{l-m}\,\phi_2(m,k)$$

are bounded. Let us consider the first summand in the decomposition for $f^{[2]}$ above. Lemma 4.2 gives

$$\frac{l-k}{l-m} = \int_{\mathbb{R}} g(s)\ (l-k)^{is}\,(l-m)^{-is}\,ds,\ \ l < k \leq m,$$

which implies that we need to study the operator

$$T_s(x,y) = \sum_{l<k\leq m} (k-l)^{is}\,(m-l)^{-is}\,\phi_2(l,k)\,E_l x E_k y E_m.$$

[4] Note that θ approaches 0 as α runs to ∞; note also that $\left\|T_{f^{[1]}}\right\|_2 \leq 1$.

If R_s is the mapping $x \mapsto x_s$ from Lemma 4.3, then, by Lemma 3.2 (iv),

$$T_s(x,y) = R_{-s}\left(T_{\phi_2}\left(R_s(x)\right)y\right).$$

Since R_s is bounded (see Lemma 4.3), T_s is also bounded, provided T_{ϕ_2} is bounded, i.e., we have reduced the question on the triple operator integral down to the question on the double one.

We shall present a detailed technical account of the scheme above in the following section. We shall use the method of mathematical induction. The base of induction is proving that the operator associated with ϕ_2, or in general, with ϕ_m, introduced below, is bounded. This part is done in Theorem 5.6. It follows the lines of the proof of Theorem 4.1, with appropriate adjustments. The step of induction is reduction as in (4.4). This part is done in Lemma 5.8. The scheme is finalised in the proof of Theorem 5.3.

5. Proof of Theorem 2.1

In this section, we prove Theorem 2.1, and this requires some preparation.

Polynomial integral momenta. Let P_n be the class of polynomials of n variables with real coefficients. Let $\kappa > 0$ and let S_n^κ be the simplex

$$S_n^\kappa = \left\{(s_0,\ldots,s_n) \in \mathbb{R}^{n+1} : \sum_{j=0}^{n} s_j = \kappa,\ s_j \geq 0,\ 0 \leq j \leq n\right\}.$$

We equip the simplex S_n^κ with the Lebesgue surface measure $d\sigma_n$ defined by

$$\int_{S_n^\kappa} \phi(s_0,\ldots,s_n)\, d\sigma_n = \int_{R_n^\kappa} \phi\left(s_0,\ldots,s_{n-1},\kappa - \sum_{j=0}^{n-1} s_j\right) dv_n, \tag{5.1}$$

for every continuous function $\phi : \mathbb{R}^{n+1} \mapsto \mathbb{C}$, where

$$R_n^\kappa = \left\{(s_0,\ldots,s_{n-1}) \in \mathbb{R}^n : \sum_{j=0}^{n-1} s_j \leq \kappa,\ s_j \geq 0,\ 0 \leq j \leq n\right\}$$

and dv_n is the Lebesgue measure on $\mathbb{R}^n$. The multiple integrals in (5.1) can be reduced to iterated integrals, which is demonstrated in the proof of Lemma 5.1 below. It can be seen via a straightforward change of variables in (5.1) that the measure $d\sigma_n$ is invariant under any permutation of the variables $s_0,\ldots,s_n$. We set $S_n := S_n^1$ and $R_n := R_n^1$.

Let

$$\tilde{s} = (s_1,\ldots,s_n) \in R_n,\ \ (s_0,\tilde{s}) \in S_n,\ \ s_0 = 1 - \sum_{j=1}^{n} s_j.$$

Given $h \in C_b$, and $p \in P_n$, we introduce

$$\phi_{n,h,p}\left(\tilde{\lambda}\right) = \int_{S_n} p\left(\tilde{s}\right)\, h\left(\sum_{j=0}^{n} s_j \lambda_j\right) d\sigma_n, \tag{5.2}$$

where $\tilde{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \mathbb{R}^{n+1}$. We shall call the function $\phi_{n,h,p}$ a *polynomial integral momentum*. The function $\phi_{n,h,p}$ is continuous since h is.

The following routine fact shows that the polynomial integral momentum is a generalization of the divided difference.

Lemma 5.1. *For $f \in C^n$, $f^{[n]} = \phi_{n,f^{(n)},1}$.*

Proof. We have that

$$\begin{aligned}
\phi_{n,f^{(n)},1} &= \int_{S_n} f^{(n)}\left(\sum_{j=0}^{n} s_j\lambda_j\right) d\sigma_n \\
&= \int_{S_n} f^{(n)}\bigg(\sum_{j=0}^{n} s_j\lambda_n + \sum_{j=0}^{n-1} s_j(\lambda_{n-1}-\lambda_n) + \cdots + \\
&\qquad\qquad + \sum_{j=0}^{1} s_j(\lambda_1-\lambda_2) + s_0(\lambda_0-\lambda_1)\bigg)\, d\sigma_n.
\end{aligned}$$

By substituting $1 = \sum_{j=0}^{n} s_j$, $t_1 = \sum_{j=0}^{n-1} s_j$, $\ldots$, $t_{n-1} = \sum_{j=0}^{1} s_j$, and $t_0 = s_0$ in the latter integral, we obtain

$$\int_0^1 dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} f^{(n)}(\lambda_n + (\lambda_{n-1}-\lambda_n)t_1 + \cdots + (\lambda_0-\lambda_1)t_n)\, dt_n,$$

which equals $f^{[n]}(\lambda_0, \lambda_1, \ldots, \lambda_n)$ by [10, Formula (7.12)]. $\square$

Lemma 5.2. *If $h \in W_0$ and $p \in P_n$, then $\phi_{n,h,p} \in \mathfrak{C}_n$ and*

$$\left\|\phi_{n,h,p}\right\|_{\mathfrak{C}_n} \leq c_{n,p}\, \|h\|_{W_0}.$$

Proof of Lemma 5.2. Since $h \in W_0$, we have $g := \hat{h} \in L^1(\mathbb{R})$, i.e.,

$$h(t) = \int_{\mathbb{R}} g(s)e^{ist}\, ds, \;\; g \in L^1(\mathbb{R}).$$

Observing that

$$h\left(\sum_{j=0}^{n} s_j\lambda_j\right) = \int_{\mathbb{R}} \prod_{j=0}^{n} e^{iss_j\lambda_j}\, g(s)\, ds,$$

implies

$$\phi_{n,h,p}\left(\tilde{\lambda}\right) = \int_{\mathbb{R}} \int_{S_n} \prod_{j=0}^{n} e^{iss_j\lambda_j}\, p(\tilde{s})\, g(s)\, d\sigma_n\, ds.$$

Thus, choosing the finite measure space $(\mathbb{R} \times S_n, g(s)\, ds \times p(\tilde{s})\, d\sigma_n)$ and functions

$$a_j(s, \tilde{s}, t) = e^{iss_jt}, \;\; 0 \leq j \leq n,$$

in (3.1), we obtain that $\phi_{n,h,p} \in \mathfrak{C}_n$ and

$$\left\|\phi_{n,h,p}\right\|_{\mathfrak{C}_n} \leq \frac{1}{n!} \, \|h\|_{W_0} \sup_{\tilde{s}\in R_n} |p(\tilde{s})| \,,$$

completing the proof. □

It follows immediately from Lemmas 3.5 and 5.2 that the multiple operator integral $T_{\phi_{n,h,p}}$ associated with the function $\phi_{n,h,p}$ is bounded on every $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, with $0 \leq \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} \leq 1$, provided $h \in W_0$. The principal step toward Theorem 2.1 is the following improvement of the observation above.

Theorem 5.3. *Let $p \in P_n$ and $h \in C_b$. Let $1 < \alpha_j < \infty$, for $1 \leq j \leq n$, be such that $0 < \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} < 1$. Then the operator $T_{\phi_{n,h,p}}$ exists and is bounded on every $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ and*

$$\left\|T_{\phi_{n,h,p}}\right\| \leq c_{n,p} \, \|h\|_{\infty} \,,$$

where the constant $c_{n,p} > 0$ depends only on the exponents α_j, $j = 1, 2, \ldots, n$, the polynomial p, and dimension n.

Remark 5.4. Theorem 5.3 and Lemma 5.1 imply that, for every $f \in C^n$, $T_{f^{[n]}}$ is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, with $0 < \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} < 1$ and

$$\left\|T_{f^{[n]}}\right\| \leq c_n \, \left\|f^{(n)}\right\|_{\infty} .$$

Theorem 5.3 and discrete spectral measures. Before proceeding to the principal part of the proof of Theorem 5.3, let us observe that essentially it needs only to be proved in the case when the operator H has discrete spectrum. As soon as that is done, the rest of the proof is a straightforward approximation. This observation is formalized in the following lemma.

Lemma 5.5. *Let $n \in \mathbb{N}$, $p \in P_n$, and let $\{E_l\}_{l\in\mathbb{Z}}$ be a sequence of pairwise orthogonal spectral projections such that $\sum_{l\in\mathbb{Z}} E_l = 1$. Let $1 < \alpha, \alpha_j < \infty$, $x \in L^{\alpha_j}$, for $1 \leq j \leq n$, and $0 < \frac{1}{\alpha} = \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n} < 1$. If, for every $h \in C_b$, the series*

$$S_h(\tilde{x}) := \sum_{l_0,\ldots,l_n\in\mathbb{Z}} \phi_{n,h,p}\left(l_0, l_1, \ldots, l_n\right) \, E_{l_0} x_1 E_{l_1} x_2 \cdot \ldots \cdot x_n E_{l_n}, \tag{5.3}$$

where $\tilde{x} = (x_1, \ldots, x_n)$, converges in L^{α}, the mapping

$$\tilde{x} \mapsto S_h(\tilde{x})$$

is bounded on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$ independently of the sequence $\{E_l\}_{l\in\mathbb{Z}}$ with the norm

$$\|S_h\| \leq c_{n,p} \, \|h\|_{\infty} \,,$$

then the claim of Theorem 5.3 holds.

Proof of Lemma 5.5. We observe that, by Lemma 3.3, it is enough to consider $\phi_{n,h,p}$ only on compact subsets of $\mathbb{R}^{n+1}$ and, therefore the sum in (5.3) is finite.

Let $h \in C_c$. Fix $n \in \mathbb{N}$, $p \in P_n$. Observe that the assumption of the lemma implies that the series

$$y_m := \sum_{l_0,\dots,l_n \in \mathbb{Z}} \phi_{n,h,p}\left(\frac{l_0}{m},\dots,\frac{l_n}{m}\right) E_{l_0,m} x_1 E_{l_1,m} x_2 \cdot \ldots \cdot x_n E_{l_n,m} \tag{5.4}$$

converges in L^α and the mappings

$$S_{h,m} : \tilde{x} \mapsto y_m$$

from Definition 3.1 are uniformly bounded polylinear operators on $L^{\alpha_1} \times \ldots \times L^{\alpha_n}$, with

$$\left\|S_{h,m}\right\| \le c_{n,p}\ \|h\|_\infty. \tag{5.5}$$

Indeed, in order to see that (5.4) fits to (5.3) we just need to take $E_l = E_{l,m}$, $l \in \mathbb{Z}$, and replace h in (5.3) with the function $h_1(t) = h\left(\frac{t}{m}\right)$, $t \in \mathbb{R}$.

Thus, we see that to finish the proof of the lemma, we have only to show that for every fixed collection $\{x_j\}_{j=0}^n \subset L^{\alpha_j}$, the sequence $\{y_m\}_{m\ge1}$ converges in L^α.

We shall show that the sequence $\{y_m\}_{m\ge1}$ is Cauchy. Fix $\epsilon > 0$ and fix a C_c^∞-function $\tilde{h}$ such that

$$\left\|h - \tilde{h}\right\|_\infty \le \frac{\epsilon}{c_{n,p}}, \tag{5.6}$$

where $c_{n,p}$ is taken from (5.5). Let $\tilde{y}_m := S_{\tilde{h},m}(\tilde{x})$. By Lemma 5.2, $\phi_{n,\tilde{h},p} \in \mathfrak{C}_n$, and, hence, Definition 3.1 and Lemma 3.5 imply that the sequence $\{\tilde{y}_m\}_{m\ge1}$ is Cauchy. That is, there is $m_\epsilon \in \mathbb{N}$ such that

$$\|\tilde{y}_m - \tilde{y}_{m'}\|_\alpha \le \epsilon, \quad \text{for } m, m' > m_\epsilon. \tag{5.7}$$

Since

$$y_m - \tilde{y}_m = S_{h-\tilde{h},m}(\tilde{x}),$$

by (5.5) and (5.6)

$$\|y_m - \tilde{y}_m\|_\alpha \le \epsilon, \quad m \ge 1.$$

Combining the latter with (5.7) implies that for every $\epsilon > 0$, there is $m_\epsilon \in \mathbb{N}$ such that for all $m, m' > m_\epsilon$,

$$\|y_m - y_{m'}\|_\alpha \le \|y_m - \tilde{y}_m\|_\alpha + \|\tilde{y}_m - \tilde{y}_{m'}\|_\alpha + \|\tilde{y}_{m'} - y_{m'}\|_\alpha \le 3\epsilon.$$

The lemma is proved. $\square$

Now we shall prove Theorem 5.3 following the scheme outlined in Section 4.

By Lemma 5.5, without loss of generality, we assume that the spectrum of the operator H is concentrated in the integral points, i.e., if dE_λ is the spectral measure of H, then

$$E(B) = \sum_{l \in B \cap \mathbb{Z}} E_l, \quad B \subseteq \mathbb{R} \text{ is Borel},$$

where $E = \{E_l\}_{l \in \mathbb{Z}}$ is a spectral family. The operator $T_{\phi_{n,h,p}}$ associated with H is given by the finite multiple sum

$$T_{\phi_{n,h,p}}(x_1, \dots, x_n) = \sum_{l_0, \dots, l_n \in \mathbb{Z}} \phi_{n,h,p}\,(l_0, \dots, l_n)\; E_{l_0} x_1 E_{l_1} x_2 \cdot \ldots \cdot x_n E_{l_n}. \tag{5.8}$$

By Lemma 3.3, it is enough to consider the polynomial integral momenta $\phi_{n,h,p}$ only on compact sets. Therefore, $T_{\phi_{n,h,p}}$ is given by the finite sum in the proofs below.

The base of induction. Note that upon integrating by parts, the function $\phi_2(\lambda, \mu)$ in (4.4) can be expressed as

$$\begin{aligned}
\phi_2(\lambda, \mu) &= \frac{f^{[1]}(\lambda, \mu) - f'(\mu)}{\lambda - \mu} \\
&= \frac{1}{\lambda - \mu} \Big(t f'(\lambda + (\mu - \lambda)t)\big|_0^1 - f'(\mu) \Big) + \int_0^1 t f''(\lambda + (\mu - \lambda)t)\, dt \\
&= \int_0^1 t f''(\lambda + (\mu - \lambda)t)\, dt.
\end{aligned}$$

The function $\phi_2(\lambda, \mu)$ is a particular case of the more general function

$$\phi_{m,h}(\lambda, \mu) = \int_0^1 t^{m-1} h(\lambda + (\mu - \lambda)t)\, dt, \tag{5.9}$$

where $h \in C_b$ and $m \in \mathbb{N}$. Note that $\phi_{m,h}$, in its turn, is the special case of $\phi_{n,h,p}$ given by (5.2) with $n = 1$ and $p(t) = t^{m-1}$. In particular, if $m = 1$, then $\phi_{1,f'} = f^{[1]}$ (see, e.g., Lemma 5.1).

The following theorem strengthens Theorem 4.1.

Theorem 5.6. *Let $h \in C_b$, $m \in \mathbb{N}$, and let $\phi_{m,h}$ be as in* (5.9). *Then the operator $T_{\phi_{m,h}}$ is bounded on every L^α, $1 < \alpha < \infty$, and*

$$\left\| T_{\phi_{m,h}} \right\|_\alpha \le c_{\alpha,m}\, \|h\|_\infty .$$

The special case $m = 1$ of Theorem 5.6 is equivalent to Theorem 4.1 with $h = f'$. For $m > 1$, the proof of Theorem 5.6 repeats the one of Theorem 4.1, with technical modifications. Recall that the key step in the proof of Theorem 4.1 is the decomposition (4.2). For Theorem 5.6, we shall need the following extension of (4.2).

Lemma 5.7. *Let $h \in C_b$, $m \in \mathbb{N}$, and let $\phi_{m,h}$ be as in* (5.9)*. Suppose that $\lambda \leq \xi \leq \mu$ and $\lambda \neq \mu$. Then*

$$\begin{aligned}\phi_{m,h}(\lambda,\mu) &= \left(\frac{\lambda-\xi}{\lambda-\mu}\right)^m \phi_{m,h}(\lambda,\xi) + \left(\frac{\xi-\mu}{\lambda-\mu}\right)^m \phi_{m,h}(\xi,\mu)\\ &\quad + \sum_{k=1}^{m-1} C_{m-1}^{k-1} \left(\frac{\lambda-\xi}{\lambda-\mu}\right)^{m-k} \left(\frac{\xi-\mu}{\lambda-\mu}\right)^k \phi_{k,h}(\xi,\mu). \end{aligned} \tag{5.10}$$

Proof of Lemma 5.7. Denote $\zeta = \frac{\lambda-\xi}{\lambda-\mu}$ and $\omega = \frac{\xi-\mu}{\lambda-\mu}$. We start with splitting

$$\phi_{m,h}(\lambda,\mu) = \int_0^1 t^{m-1} h(\lambda + (\mu-\lambda)t)\, dt = \int_0^\zeta + \int_\zeta^1. \tag{5.11}$$

We compute these integrals separately. Substituting $t = \zeta t_1$ in the first integral gives

$$\int_0^\zeta = \int_0^1 \zeta^{m-1} t_1^{m-1} h(\lambda + (\xi-\lambda)t_1)\, \zeta dt_1 = \zeta^m \phi_{m,h}(\lambda,\xi). \tag{5.12}$$

Substituting $t = \zeta + \omega t_2$ in the second integral and using the Newton binomial formula gives

$$\begin{aligned}\int_\zeta^1 &= \int_0^1 (\zeta+\omega t_2)^{m-1} h(\xi + (\mu-\xi)t_2)\, \omega dt_2\\ &= \int_0^1 \omega \left[\sum_{k=0}^{m-1} C_{m-1}^k \zeta^{m-k-1} \omega^k t_2^k\right] h(\xi + (\mu-\xi)t_2)\, dt_2\\ &= \int_0^1 \left[\omega^m t_2^{m-1} + \sum_{k=1}^{m-1} C_{m-1}^{k-1} \zeta^{m-k} \omega^k t_2^{k-1}\right] h(\xi + (\mu-\xi)t_2)\, dt_2\\ &= \omega^m \phi_{m,h}(\xi,\mu) + \sum_{k=1}^{m-1} C_{m-1}^{k-1} \zeta^{m-k} \omega^k \phi_{k,h}(\xi,\mu). \end{aligned} \tag{5.13}$$

Combining (5.11) - (5.13) gives (5.10). $\square$

Now we prove Theorem 5.6.

Proof of Theorem 5.6. It is sufficient to prove the theorem for real-valued $h \in C_b$ such that $\|h\|_\infty \leq 1$.

Our objective is to show existence of a constant $c_{\alpha,m} > 0$ such that

$$\left|\tau\left(yT_{\phi_{m,h}}(x)\right)\right| \leq c_{\alpha,m}\left(1 + \left\|T_{\phi_{m,h}}\right\|_\beta\right) \tag{5.14}$$

for $x \in L^\alpha$ and $y \in L^{\alpha'}$ with $\|x\|_\alpha = 1$ and $\|y\|_{\alpha'} = 1$, where $\frac{1}{2} = \frac{1}{\alpha} + \frac{1}{\beta}$. As in the case of Lemma 4.4, it is enough to prove (5.14) for x being off-diagonal upper-triangular and y being lower-triangular with τ-finite left and right supports. If (5.14) is proved, then the remaining argument is similar to the verbatim repetition of the extrapolation trick in the proof of Theorem 4.1. The proof of (5.14) is

(only) computationally more difficult argument than the one in Lemma 4.4, where the relation (4.2) is replaced with (5.10).

Fix $\epsilon > 0$ and factorize $y = ab$, where $a \in L^2$ and $b \in L^\beta$ are lower-triangular such that

$$1 \leq \|a\|_2 \ \|b\|_\beta \leq 1 + \epsilon.$$

Keeping the same notation $z_{\lambda\mu} = E_\lambda z E_\mu$, for every $z \in M$, and arguing exactly as in Lemma 4.4, we have

$$\tau\left(yT_{\phi_{m,h}}(x)\right) = \tau\left(abT_{\phi_{m,h}}(x)\right) = \sum_{\substack{\lambda\leq\xi\leq\mu \\ \lambda\neq\mu}} \phi_{m,h}(\lambda,\mu)\,\tau\left(a_{\mu\xi}b_{\xi\lambda}x_{\lambda\mu}\right).$$

Let g be the function from Lemma 4.2. Denote $x_s := \sum_{\lambda<\mu}(\mu-\lambda)^{is}x_{\lambda\mu}$ and $b_s := \sum_{\lambda<\xi}(\xi-\lambda)^{is}b_{\xi\lambda}$. Applying Lemma 3.2 and the representation (5.10) yields

$$\begin{aligned}\tau\left(yT_{\phi_{m,h}}(x)\right) = \int_{\mathbb{R}} g(s)\Bigg[&\tau\left(a\left(T_{\bar{\phi}_{m,h}}(b)\right)_{ms}x_{-ms}\right) + \tau\left(\left(T_{\bar{\phi}_{m,h}}(a)\right)_{ms}bx_{-ms}\right) \\ &+ \sum_{k=1}^{m-1} C_{m-1}^{k-1}\tau\left(T_{\phi_{k,h}}\left(a_{ks}\right)b_{ms-ks}x_{-ms}\right)\Bigg]\,ds, \quad (5.15)\end{aligned}$$

where $\bar{\phi}_{m,h}(\lambda,\mu) := \phi_{m,h}(\mu,\lambda)$. First, we estimate the integrand components. By employing Remark 3.4 and the representation (5.11) and recalling $1 \leq k \leq m$,

$$\|T_{\phi_{k,h}}(a_{ks})\|_2 \leq \|\phi_{k,h}\|_\infty\|a_{ks}\|_2 \leq \|h\|_\infty\|a\|_2 \leq \|a\|_2,$$

we obtain

$$\left|\tau\left(T_{\phi_{k,h}}\left(a_{ks}\right)b_{ms-ks}x_{-ms}\right)\right| \leq c_\alpha\|a\|_2\|b_{ms-ks}\|_\beta\|x_{-ms}\|_\alpha \leq c_\alpha(1+|ms|)^2(1+\epsilon).$$

Note that, by Lemma 3.2 (i), we have $\left\|T_{\phi_{m,h}}\right\| = \left\|T_{\bar{\phi}_{m,h}}\right\|$. Arguing as in Lemma 4.4 implies

$$\begin{aligned}\left|\tau\left(a\left(T_{\bar{\phi}_{m,h}}(b)\right)_{ms}x_{-ms}\right)\right| &\leq \|a\|_2\ \left\|\left(T_{\bar{\phi}_{m,h}}(b)\right)_{ms}\right\|_\beta \|x_{-ms}\|_\alpha \\ &\leq\ c_\alpha\,(1+|ms|)^2\ \|a\|_2\left\|T_{\bar{\phi}_{m,h}}(b)\right\|_\beta\|x\|_\alpha \\ &\leq\ c_\alpha\,(1+|ms|)^2\ \left\|T_{\phi_{m,h}}\right\|_\beta\ \|a\|_2\ \|b\|_\beta \\ &\leq\ c_\alpha\,(1+|ms|)^2\ \left\|T_{\phi_{m,h}}\right\|_\beta(1+\epsilon).\end{aligned}$$

By letting $\epsilon \to 0$, we arrive at

$$\left|\tau\left(a\left(T_{\bar{\phi}_{m,h}}(b)\right)_{ms}x_{-ms}\right)\right| \leq c_\alpha\,(1+|ms|)^2\ \left\|T_{\phi_{m,h}}\right\|_\beta.$$

Similarly,

$$\left|\tau\left(\left(T_{\bar{\phi}_{m,h}}(a)\right)_{ms}bx_{-ms}\right)\right| \leq c_\alpha\ (1+|ms|)^2.$$

Employing the triangle inequality in (5.15), we see that

$$\begin{aligned}\left|\tau\left(abT_{\phi_{m,h}}(x)\right)\right| &\leq c_\alpha\left(1+\left\|T_{\phi_{m,h}}\right\|_\beta\right)\int_{\mathbb{R}}|g(s)|\,(1+|ms|)^2\,ds\\&\leq c_{\alpha,m}\left(1+\left\|T_{\phi_{m,h}}\right\|_\beta\right).\end{aligned}$$

This proves (5.14) and, hence, completes the proof of the theorem. □

The induction step. We need an extension of the decomposition (4.4) to the case of higher dimensions. Let

$$\tilde{s}=(s_1,\dots,s_n)\in R_n,\ \ (s_0,\tilde{s})\in S_n,\quad \text{with } s_0=\kappa-\sum_{j=1}^n s_j.$$

Given $h \in C_b$, and $p \in P_{n+1}$, we introduce

$$\psi_{n,h,p}(\zeta,\tilde{\mu})=\int_{S_n}p(\zeta,\tilde{s})\,h\left(\sum_{j=0}^n s_j\mu_j\right)d\sigma_n, \tag{5.16}$$

where $\zeta\in\mathbb{R}$, $\tilde{\mu}=(\mu_0,\dots,\mu_n)\in\mathbb{R}^{n+1}$.

Lemma 5.8. *Let $n\geq 2$, $h\in C_b$ and let $p\in P_n$. Denote $\tilde{\lambda}=(\lambda_3,\dots,\lambda_n)\in\mathbb{R}^{n-2}$ and assume that $\lambda_0\leq\lambda_2\leq\lambda_1$, with $\lambda_0\neq\lambda_1$. Then there are polynomials $q,r\in P_n$ depending only on p such that the function $\phi_{n,h,p}$ given by* (5.2) *equals*

$$\begin{aligned}&\phi_{n,h,p}\left(\lambda_0,\lambda_1,\lambda_2,\tilde{\lambda}\right)\\&\quad=\psi_{n-1,h,q}\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\lambda_0,\lambda_2,\tilde{\lambda}\right)+\psi_{n-1,h,r}\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\lambda_1,\lambda_2,\tilde{\lambda}\right).\end{aligned} \tag{5.17}$$

We prove Lemma 5.8 by reducing it to the special case $n=2$ discussed in the lemma below.

Lemma 5.9. *Let $h\in C_b$ and let $m,k\in\mathbb{N}\cup\{0\}$. Let $\kappa>0$ and $\lambda\leq\xi\leq\mu$, with $\lambda\neq\mu$. Then there are $q,r\in P_3$ such that*

$$\begin{aligned}&\int_0^\kappa t^m dt\int_0^t s^k h(\kappa\xi+(\lambda-\xi)t+(\mu-\lambda)s)\,ds\\&\qquad=\int_0^\kappa q\left(\frac{\lambda-\xi}{\lambda-\mu},\kappa,\theta\right)h(\kappa\xi+(\lambda-\xi)\theta)\,d\theta\\&\qquad\quad+\int_0^\kappa r\left(\frac{\lambda-\xi}{\lambda-\mu},\kappa,\sigma\right)h(\kappa\xi+(\mu-\xi)\sigma)\,d\sigma.\end{aligned}$$

Here the polynomials q and r depend on m and k, but do not depend on h.

First, we prove the decomposition (5.17) and, then, auxiliary Lemma 5.9.

Proof of Lemma 5.8. By the definitions of the functions $\phi_{n,h,p}$ and $\psi_{n-1,h,q}$, we have integration over the simplex S_n (for which $\sum_{j=0}^{n} s_j = 1$) on both sides of (5.17). Since $\sum_{j=0}^{2} s_j = 1 - \sum_{j=3}^{n} s_j$, we can split the integral over R_n (which is defined on p. 15) into the repeated integral

$$\int_{R_{n-2}} ds_3 \dots ds_n \int_{S_2^\kappa} ds_2 ds_1 ds_0, \tag{5.18}$$

where $\kappa = 1 - \sum_{j=3}^{n} s_j$. We fix the point $\tilde{s} = (s_3, \dots, s_n) \in R_{n-2}$ (and, hence, fix κ too) and consider the integrands over S_2^κ. On the left hand side, we have the integrand

$$\int_{S_2^\kappa} p_1(s_0, s_1, s_2) h_1 \left(s_0\lambda_0 + s_1\lambda_1 + s_2\lambda_2\right) d\sigma_2, \tag{5.19}$$

where we set

$$p_1(s_0, s_1, s_2) := p(s_1, s_2, s_2, \tilde{s}), \;\; h_1(t) := h\left(t + \sum_{j=3}^{n} s_j\lambda_j\right).$$

Note that

$$s_0\lambda_0 + s_1\lambda_1 + s_2\lambda_2 = (s_0 + s_1 + s_2)\lambda_2 + (s_0 + s_1)(\lambda_0 - \lambda_2) + s_1(\lambda_1 - \lambda_0).$$

Further, by recalling $\kappa = s_0 + s_1 + s_2$, making substitution $t = s_0 + s_1$, and $s = s_1$ and setting

$$\lambda_0 = \lambda, \quad \lambda_1 = \mu, \quad \lambda_2 = \xi, \tag{5.20}$$

we see that (5.19) becomes

$$\int_0^\kappa dt \int_0^t p_2(t,s)\, h_1(\kappa\xi + (\lambda - \xi)t + (\mu - \lambda)s)\, ds, \tag{5.21}$$

where

$$p_2(t,s) = p_1(t - s, s, \kappa - t).$$

Note that p_2 is a polynomial of two variables t and s, that is, it is a finite linear combination of monomials $t^m s^k$ multiplied by certain powers of fixed $\tilde{s}$ (these powers are determined by p). Consequently, applying Lemma 5.9, we see that (5.21) equals

$$\int_0^\kappa q_1(\zeta, \kappa, \theta) h_1(\kappa\xi + (\lambda - \xi)\theta)\, d\theta + \int_0^\kappa r_1(\zeta, \kappa, \sigma) h_1(\kappa\xi + (\mu - \xi)\sigma)\, d\sigma, \tag{5.22}$$

where $\zeta = \frac{\lambda - \xi}{\lambda - \mu}$ and $q_1, r_1 \in P_3$ are appropriate linear combinations of the polynomials from Lemma 5.9 (the precise representation for which is given in (5.25) below). If we substitute now $s_0 = \theta$, $s_2 = \kappa - \theta$ in the first integral and $s_1 = \sigma$,

$s_2 = \kappa - \sigma$ in the second one, and recall the equalities (5.20), then we see that (5.22) equals

$$\int_{S_1^\kappa} q_1\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\kappa,s_0\right) h_1\left(s_0\lambda_0+s_2\lambda_2\right) d\sigma_1 \\ + \int_{S_1^\kappa} r_1\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\kappa,s_1\right) h_1\left(s_1\lambda_1+s_2\lambda_2\right) d\sigma_1.$$

Recall that p_2 was obtained from p by fixing the value of $\tilde{s}$. By letting $\tilde{s}$ vary, we obtain from q_1 and r_1 in P_3 such polynomials q and r in P_n that

$$\int_{S_2^\kappa} p(s_0,s_1,s_2,\tilde{s})h\left(s_0\lambda_0+s_1\lambda_1+s_2\lambda_2+\sum_{j=3}^n s_j\lambda_j\right) d\sigma_2 \\ = \int_{S_1^\kappa} q\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\kappa,s_0,\tilde{s}\right) h\left(s_0\lambda_0+s_2\lambda_2+\sum_{j=3}^n s_j\lambda_j\right) d\sigma_1 \\ + \int_{S_1^\kappa} r\left(\frac{\lambda_0-\lambda_2}{\lambda_0-\lambda_1},\kappa,s_1,\tilde{s}\right) h\left(s_1\lambda_1+s_2\lambda_2+\sum_{j=3}^n s_j\lambda_j\right) d\sigma_1.$$

Now we integrate the latter expression over R_{n-2} with respect to $\tilde{s}$ and obtain (5.17). $\square$

Poof of Lemma 5.9. We first prove the case $\kappa = 1$. Let us compute the integral on the left hand side

$$\text{LHS} := \int_0^1 t^m dt \int_0^t s^k h(\xi+(\lambda-\xi)t+(\mu-\lambda)s)\, ds \tag{5.23}$$

by substituting

$$u = \xi+(\lambda-\xi)t+(\mu-\lambda)s.$$

We have

$$\text{LHS} = \int_0^1 t^m dt \int_{\xi+(\lambda-\xi)t}^{\xi+(\mu-\xi)t} \left[\frac{u-\xi-(\lambda-\xi)t}{\mu-\lambda}\right]^k \frac{h(u)\,du}{\mu-\lambda}.$$

Next, we observe that changing the order of integration yields

$$\int_0^1 dt \int_{\xi+(\lambda-\xi)t}^{\xi+(\mu-\xi)t} du = \int_\lambda^\xi du \int_{\frac{u-\xi}{\lambda-\xi}}^1 dt + \int_\xi^\mu du \int_{\frac{u-\xi}{\mu-\xi}}^1 dt.$$

With this change of the order, we obtain

$$\text{LHS} = \int_\lambda^\xi \frac{h(u)\,du}{\mu-\lambda} \int_{\frac{u-\xi}{\lambda-\xi}}^1 \left[\frac{u-\xi-(\lambda-\xi)t}{\mu-\lambda}\right]^k t^m\, dt \\ + \int_\xi^\mu \frac{h(u)\,du}{\mu-\lambda} \int_{\frac{u-\xi}{\mu-\xi}}^1 \left[\frac{u-\xi-(\lambda-\xi)t}{\mu-\lambda}\right]^k t^m\, dt.$$

Further, we substitute

$$\zeta = \frac{\lambda-\xi}{\lambda-\mu}, \quad \theta = \frac{u-\xi}{\lambda-\xi}, \quad \text{and} \quad \sigma = \frac{u-\xi}{\mu-\xi}$$

in the first and in the second integrals, respectively. This gives

$$\text{LHS} = \zeta \int_0^1 h(\xi + (\lambda - \xi)\theta)\, d\theta \int_\theta^1 \zeta^k (t-\theta)^k t^m\, dt$$
$$+ (1-\zeta) \int_0^1 h(\xi + (\mu - \xi)\sigma)\, d\sigma \int_\sigma^1 ((1-\zeta)\sigma - \zeta t)^k t^m\, dt. \quad (5.24)$$

Setting

$$q(\zeta,\theta) := \zeta^{k+1} \int_\theta^1 (t-\theta)^k t^m dt \quad \text{and}$$
$$r(\zeta,\sigma) := (1-\zeta) \int_\sigma^1 ((1-\zeta)\,\sigma - \zeta t)^k t^m\, dt$$

and observing that $q, r \in P_2$ finishes the proof when $\kappa = 1$.

In order to see the case with arbitrary $\kappa > 0$, we replace λ, μ, and ξ in (5.23) with $\kappa\lambda_1$, $\kappa\mu_1$, and $\kappa\xi_1$ and substitute

$$t_1 = \kappa t,\ \ s_1 = \kappa s,\ \ \theta_1 = \kappa\theta,\ \text{ and }\ \sigma_1 = \kappa\sigma.$$

This gives

$$\int_0^\kappa \frac{t_1^m dt_1}{\kappa^{m+1}} \int_0^{t_1} h(\kappa\xi_1 + (\lambda_1 - \xi_1)t_1 + (\mu_1 - \lambda_1)s_1)\, \frac{s_1^k ds_1}{\kappa^{k+1}}$$
$$= \zeta^{k+1} \int_0^\kappa h(\kappa\xi_1 + (\lambda_1 - \xi_1)\theta_1)\, \frac{d\theta_1}{\kappa} \int_{\theta_1}^\kappa (t_1 - \theta_1)^k t_1^m \frac{dt_1}{\kappa^{k+m+1}}$$
$$+ (1-\zeta) \int_0^\kappa h(\kappa\xi_1 + (\mu_1 - \xi_1)\sigma_1)\, \frac{d\sigma_1}{\kappa} \int_{\sigma_1}^\kappa ((1-\zeta)\sigma_1 - \zeta t_1)^k\, t_1^m \frac{dt_1}{\kappa^{k+m+1}}.$$

We cancel the factor κ^{-m-k-2} on both sides of the previous calculation and set

$$q(\zeta,\kappa,\theta_1) := \zeta^{k+1} \int_{\theta_1}^\kappa (t_1 - \theta_1)^k t_1^m dt_1 \quad \text{and}$$
$$r(\zeta,\kappa,\sigma_1) := (1-\zeta) \int_{\sigma_1}^\kappa ((1-\zeta)\sigma_1 - \zeta t_1)^k\, t_1^m dt_1. \quad (5.25)$$

Observing $q, r \in P_3$ finishes the proof in the general case $\kappa > 0$. □

We are now ready to complete the proof of Theorem 5.3.

Proof of Theorem 5.3. It is sufficient to prove the theorem for real-valued $h \in C_b$. It is also sufficient to prove the theorem when H has spectrum at integral points (see Lemma 5.5) and consider $\phi_{n,h,p}$ only on compact sets (see Lemma 3.3).

The proof is by the method of the mathematical induction with respect to $n \in \mathbb{N}$. The base of the induction, i.e., the case $n = 1$, is done in Theorem 5.6.

Let us assume that $n > 1$ and that the theorem is proved for $n - 1$, that is, we have

$$\left\| T_{\phi_{n-1,h,p}} \right\| \le c_{n-1,p}\ \|h\|_\infty. \quad (5.26)$$

Let $1 < \alpha_0 < \infty$ be such that $1 = \frac{1}{\alpha_0} + \frac{1}{\alpha_1} + \ldots + \frac{1}{\alpha_n}$ and let $x_j \in L^{\alpha_j}$ be such that $\left\|x_j\right\|_{\alpha_j} = 1, 0 \leq j \leq n$. We shall show that

$$\sup_{x_j \in L^{\alpha_j},\ \|x_j\|_{\alpha_j} \leq 1} \left|\tau\left(x_0 T_{\phi_{n,h,p}}\left(\tilde{x}\right)\right)\right| \leq c_{n,p}. \tag{5.27}$$

Recall that (see (5.8))

$$\tau\left(x_0 T_{\phi_{n,h,p}}\left(\tilde{x}\right)\right) = \sum_{\tilde{l} \in \mathbb{Z}^{n+1}} \phi_{n,h,p}\left(\tilde{l}\right) \tau\left(\prod_{j=0}^{n} x_j E_{l_j}\right), \ \ \tilde{l} = \left(l_0, l_1, \ldots, l_n\right).$$

Let $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, $\epsilon_j = \pm 1$ and let $K_\epsilon \subseteq \mathbb{Z}^{n+1}$ be such that

$$K_\epsilon = \Big\{(l_0, \ldots, l_n) \in \mathbb{Z}^{n+1} :\ \ l_{j-1} \leq l_j \text{ if } \epsilon_j = 1;\ \ l_{j-1} > l_j \text{ if } \epsilon_j = -1,\ \ 1 \leq j \leq n\Big\}.$$

The space $\mathbb{Z}^{n+1}$ splits into the disjoint union of 2^n sets K_ϵ, $\epsilon \in \{-1,1\}^n$. Fix ϵ. There is an index $0 \leq j_\epsilon \leq n$ such that

$$(l_0, \ldots, l_n) \in K_\epsilon \implies l_{j_\epsilon - 1} \leq l_{j_\epsilon} \text{ and } l_{j_\epsilon} > l_{j_\epsilon + 1}, \tag{5.28}$$

where the decrement and increment of the indices $j_\epsilon - 1$ and $j_\epsilon + 1$ are understood modulo n, i.e., if $j = 0$, then $j - 1 = n$ and if $j = n$, then $j + 1 = 0$. Next, by fixing j_ϵ, we further split K_ϵ into subsets $K_{\epsilon,i}$, $i = 0, 1$, where

$$K_{\epsilon,0} = \left\{(l_0, \ldots, l_n) \in K_\epsilon,\ \ l_{j_\epsilon - 1} \leq l_{j_\epsilon + 1}\right\} \quad \text{and}$$
$$K_{\epsilon,1} = \left\{(l_0, \ldots, l_n) \in K_\epsilon,\ \ l_{j_\epsilon - 1} > l_{j_\epsilon + 1}\right\},$$

The space $\mathbb{Z}^{n+1}$ splits into the disjoint union of 2^{n+1} sets $K_{\epsilon,i}$, i.e.,

$$\mathbb{Z}^{n+1} = \bigcup_{\epsilon \in \{-1,1\}^n} \bigcup_{i=0,1} K_{\epsilon,i}.$$

This means that

$$T_{\phi_{n,h,p}} = \sum_{\epsilon} \sum_{i=0,1} T^{\epsilon,i}_{\phi_{n,h,p}},$$

where

$$T^{\epsilon,i}_{\phi_{n,h,p}}(\tilde{x}) = \sum_{\tilde{l} \in K_{\epsilon,i}} \phi_{n,h,p}\left(\tilde{l}\right) \ E_{l_0} x_1 E_{l_1} x_2 \ldots x_n E_{l_n}.$$

We show (5.27) for each $T^{\epsilon,i}_{\phi_{n,h,p}}$. This will finish the proof of the theorem.

We fix $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1,1\}^n$, fix the index $j_\epsilon \in \{0, 1, \ldots, n\}$ as in (5.28), and let $i = 0, 1$. We then have

$$(l_0, l_1, \ldots, l_n) \in K_{\epsilon,i} \implies \begin{cases} l_{j_\epsilon - 1} \leq l_{j_\epsilon + 1} < l_{j_\epsilon} & \text{if } i = 0, \\ l_{j_\epsilon + 1} < l_{j_\epsilon - 1} \leq l_{j_\epsilon} & \text{if } i = 1. \end{cases}$$

By shifting and reversing if $i = 1$ (as in Lemma 3.2 (i) and (ii)) the enumeration of the variables l_j and operators x_j, we may assume that

$$l_0 \leq l_2 < l_1.$$

Now we apply Lemma 5.8. Let $q, r \in P_n$ be such that

$$\phi_{n,h,p}(l_0, l_1, l_2, \ldots, l_n) = \psi_{n-1,h,r}(\zeta, l_1, l_2, \ldots, l_n) + \psi_{n-1,h,q}(\zeta, l_0, l_2, \ldots, l_n),$$

where $\zeta = \frac{l_0 - l_2}{l_0 - l_1}$. Let $Q_{n-1,h,r}$ and $R_{n-1,h,q}$ be the multiple operator integrals corresponding to the right hand side functions $\psi_{n-1,h,r}$ and $\psi_{n-1,h,q}$. From the above decomposition we obtain

$$T^{\epsilon,i}_{\phi_{n,h,p}} = Q_{n-1,h,r} + R_{n-1,h,q}.$$

We shall show that the estimate (5.27) holds for both $Q_{n-1,h,r}$ and $R_{n-1,h,q}$. We show it for $Q_{n-1,h,r}$; for $R_{n-1,h,q}$ the proof is similar.

Let $\tilde{s} = (s_1, \ldots, s_n) \in S_{n-1}$. Since the polynomial $r(\zeta, \tilde{s})$ is a linear combination of the polynomials $r_1(\zeta, \tilde{s}) = (1 - \zeta)^m p_1(\tilde{s})$, with $m \geq 0$ and $p_1 \in P_{n-1}$ (see the integration in (5.25)), it is sufficient to prove (5.27) for Q_{n-1,h,r_1}, i.e., where r in $Q_{n-1,h,r}$ is replaced with r_1.

From (5.2) and (5.16), we have

$$\psi_{n-1,h,r_1} = (1 - \zeta)^m \phi_{n-1,h,p_1}.$$

Let g be as in Lemma 4.2 and let, as in Lemma 4.3,

$$x_{s,1} = \sum_{l_0 < l_1} (l_1 - l_0)^{is} E_{l_0} x_1 E_{l_1} \quad \text{and} \quad x_{s,2} = \sum_{l_2 < l_1} (l_1 - l_2)^{is} E_{l_2} x_2 E_{l_1}.$$

By Lemma 4.2,

$$(1 - \zeta)^m = \left(\frac{l_2 - l_1}{l_0 - l_1} \right)^m = \int_{\mathbb{R}} g(s) \, (l_1 - l_2)^{ms} \, (l_1 - l_0)^{-ms} \, ds.$$

Applying (iii) and (iv) of Lemma 3.2 gives the reduction

$$Q_{n-1,h,r_1}(x_1, x_2, \tilde{x}) = \int_{\mathbb{R}} g(s) \, x_{-ms,1} \, T_{\phi_{n-1,h,p_1}}(x_{ms,2}, \tilde{x}) \, ds, \;\; \tilde{x} = (x_3, \ldots, x_n),$$

of the problem of order n to the problem of order $n - 1$, i.e., Q_{n-1,h,r_1} is the integral of order n and $T_{\phi_{n-1,h,p_1}}$ is the integral of order $n - 1$.

Recall that by the assumption of the induction, the operator $T_{\phi_{n-1,h,p_1}}$ is bounded (see (5.26)). Recall also that by Lemma 4.3,

$$\|x_{-ms,1}\|_{\alpha_1} \leq c_{\alpha_1} \, (1 + |ms|) \quad \text{and} \quad \|x_{ms,2}\|_{\alpha_2} \leq c_{\alpha_2} \, (1 + |ms|)^2.$$

Thus,

$$\left\|Q_{n-1,h,r_1}\right\| \le c_{p_1,\alpha_1,\dots,\alpha_n} \int_{\mathbb{R}} |g(s)|\ (1+|ms|)^2\ ds \cdot \|h\|_\infty \le c_{m,p_1,\alpha_1,\dots,\alpha_n} \cdot \|h\|_\infty,$$

The latter justifies (5.27) for Q_{n-1,h,r_1} and, thus, finishes the proof. $\square$

We conclude our exposition with the proof of Theorem 2.1.

Proof of Theorem 2.1. The facts $\frac{d^n}{dt^n}\left[f\left(H_t\right)\right] \in \mathcal{L}^1$ and $\Delta_{n,f}(H,V) \in \mathcal{L}^1$ can be derived from [2,20]. We prove the second fact; the first one is even simpler. From [24, Theorem 1.43 and 1.45], we have

$$\Delta_{n,f}(H,V) = \frac{1}{(n-1)!}\int_0^1 (1-t)^{n-1}\frac{d^n}{dt^n}\left[f\left(H_t\right)\right]\,dt, \tag{5.29}$$

where, by [2, Theorem 5.7], the operator derivative is well defined and equals

$$\frac{d^n}{dt^n}\left[f(H_t)\right] = n!\,T_{t,f^{[n]}}(\underbrace{V,\dots,V}_{n\text{-times}}). \tag{5.30}$$

For the operator $T_{t,f^{[n]}}$, by Lemmas 3.5, 5.1, and 5.2, we have $\left\|T_{t,f^{[n]}}\right\| \le \|f\|_{W_n}$ and, therefore,

$$\Delta_{n,f}(H,V) \in \mathcal{L}^1 \ \text{ and } \ \left\|\Delta_{n,f}(H,V)\right\|_{\mathcal{L}_1} \le \|f\|_{W_n}\,\|V\|_n^n. \tag{5.31}$$

We shall now justify the estimate (2.2). Using (5.30) in (5.29), we see that

$$\Delta_{n,f}(H,V) = n\int_0^1 (1-t)^{n-1} T_{t,f^{[n]}}(\underbrace{V,\dots,V}_{n\text{-times}})\,dt.$$

By [2, Lemma 3.10], we have

$$\tau\left(\Delta_{n,f}(H,V)\right) = n\int_0^1 (1-t)^{n-1}\tau\left(T_{t,f^{[n]}}(\underbrace{V,\dots,V}_{n\text{-times}})\right)\,dt. \tag{5.32}$$

Thus, to show (2.2), it is sufficient to see that there is a constant $c_n > 0$ such that

$$\left|\tau\left(T_{t,f^{[n]}}(\underbrace{V,\dots,V}_{n\text{-times}})\right)\right| \le c_n\,\left\|f^{(n)}\right\|_\infty \|V\|_n^n,\ \ 0 \le t \le 1. \tag{5.33}$$

One can derive from the representation (3.2) for $T_{f^{[n]}}$ that

$$\tau\left(T_{t,f^{[n]}}(\underbrace{V,\dots,V}_{n\text{-times}})\right) = \tau\left(T_{t,\phi}(\underbrace{V,\dots,V}_{n-1\text{-times}})\,V\right), \tag{5.34}$$

where $T_{t,\phi}$ is the multiple operator integral associated with H_t and

$$\phi(\lambda_0,\lambda_1,\dots,\lambda_{n-1}) = f^{[n]}\left(\lambda_0,\lambda_0,\lambda_1,\dots,\lambda_{n-1}\right)$$

(for more details, see, e.g., [26, Lemma 3.8]). Noting that ϕ is $\phi_{n-1,h,p}$ with $h = f^{(n)}$ and $p(s_1, \ldots, s_{n-1}) = 1 - \sum_{j=1}^{n-1} s_j$, from Theorem 5.3, we obtain

$$\Big\| T_{t,\phi}(\underbrace{V, \ldots, V}_{n-1\text{-times}}) \Big\|_{\frac{n}{n-1}} \leq c_n \left\| f^{(n)} \right\|_\infty \|V\|_n^{n-1}, \ 0 \leq t \leq 1. \tag{5.35}$$

Clearly, (5.32) and (5.35) combined with (5.34) imply (5.33). Thus, (2.2) follows. To prove the continuity of $V \mapsto \tau\left(\Delta_{n,f}(H,V)\right)$, we notice that

$$\begin{aligned} &\left|\tau\left(\Delta_{n,f}(H,V_j) - \Delta_{n,f}(H,V_k)\right)\right| \\ &= \left| \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} \tau\left(T_{t,f^{[n]}}(\underbrace{V_j, \ldots, V_j}_{n\text{-times}}) - T_{t,f^{[n]}}(\underbrace{V_k, \ldots, V_k}_{n\text{-times}}) \right) dt \right|. \end{aligned} \tag{5.36}$$

Similarly to (5.33), it follows from Theorem 5.3 that

$$\left|\tau\left(T_{t,f^{[n]}}(x_1, \ldots, x_n)\right)\right| \leq c_n \|x_1\|_n \cdot \ldots \cdot \|x_n\|_n, \ 0 \leq t \leq 1, \tag{5.37}$$

for every $x_1, \ldots, x_n \in L^n$ and for every $f \in \left(\cap_{k=0}^n W_k\right) \cap B$. Combining (5.36) and (5.37) completes the proof. $\square$

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